ON THE EXISTENCE OF MAXIMIZING CURVES FOR THE CHARGED-PARTICLE ACTION

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Abstract. The classical Avez-Seifert theorem is generalized to the case of the Lorentz force equation for charged test particles with fixed charge-to-mass ratio. Given two events x_0 and x_1, with x_1 in the chronological future of x_0, and a ratio q/m, it is proved that a timelike connecting solution of the Lorentz force equation exists provided there is no null connecting geodesic and the spacetime is globally hyperbolic. As a result, the theorem answers affirmatively to the existence of timelike connecting solutions for the particular case of Minkowski spacetime. Moreover, it is proved that there is at least one C^1 connecting curve that maximizes the functional $I[\gamma] = \int_\gamma \mathrm{d}s + q/(mc^2)\omega$ over the set of C^1 future-directed non-spacelike connecting curves.

1. Introduction

Let Λ be a Lorentzian manifold endowed with the metric g having signature $(+\,-\,-\,-)$. A point particle of rest mass m and electric charge q, moving in the electromagnetic field F, has a timelike worldline that satisfies the *Lorentz force equation* (cf. [8])

$$D_s\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right) = \frac{q}{mc^2}\hat{F}(x)\left[\frac{\mathrm{d}x}{\mathrm{d}s}\right]. \tag{1}$$

Here $x = x(s)$ is the world line of the particle parameterized with proper time, $\frac{\mathrm{d}\,x}{\mathrm{d}\,s}$ is the four-velocity, $D_s\left(\frac{\mathrm{d}\,x}{\mathrm{d}\,s}\right)$ is the covariant derivative of $\frac{\mathrm{d}\,x}{\mathrm{d}\,s}$ along $x(s)$ associated to the Levi-Civita connection of g, and $\hat{F}(x)[\cdot]$ is the linear map on $T_x\Lambda$ defined by

$$g(x)[v, \hat{F}(x)[w]] = F(x)[v, w],$$

for any $v,\ w \in T_x\Lambda$.

Let x_0 and x_1 be two chronologically related events, $x_1 \in I^+(x_0)$. If the manifold Λ is globally hyperbolic, the Avez-Seifert theorem [1, 2, 3] assures the existence of at least a timelike connecting solution of the Lorentz force equation in the $q = 0$ case. We are looking for a suitable generalization to $q/m \neq 0$ cases.

Works in this direction [5, 4] have shown that in a globally hyperbolic manifold Λ, and for an exact electromagnetic field $F = d\omega$ (i.e. in absence of monopoles), connecting solutions exist for any ratio q/m in a suitable neighborhood $[-R, R]$. R is a gauge invariant quantity that depends on the extremals x_0 and x_1 and on the potential one-form. That result was satisfying from the physical point of view since for sufficiently weak field, compatible with the absence of quantum pair creation effects, the electron's charge-to-mass ratio is less than R.

From a mathematical point of view, however, the problem in the strong field case was still open. Here we prove that under the same conditions as above $R = +\infty$ provided there is no null connecting geodesic.

Like in previous papers on the subject [4, 5], the strategy is to introduce a Kaluza-Klein spacetime [9, 7] and to regard the solutions of the Lorentz force equation as projections of null geodesics of a higher dimensional manifold. In this way one can take advantage of causal techniques. Here the reference text for most notations and results on causal techniques is [6].

So assume that F is an exact two-form and let ω be a potential one-form for F. Let us consider a trivial bundle $P = \Lambda \times \mathbb{R}$, $\pi : P \to \Lambda$, with the structure group $T_1 : b \in T_1$, $p = (x, y)$, $p' = pb = (x, y + b)$, and $\tilde{\omega}$ the connection one-form on P:

$$\tilde{\omega} = i(\mathrm{d}y + \frac{e}{\hbar c}\omega).$$

Here y is a dimensionless coordinate on the fibre, $-e\,(e > 0)$ is the electron charge and $\hbar = h/2\pi$, with h the Planck constant. Henceforth we will denote by $\bar{\omega}$ and $\bar{F}$, respectively the one-form $\frac{e}{\hbar c}\omega$ and the two-form $\frac{e}{\hbar c}F$. Let us endow P with the Kaluza-Klein metric

$$g^{\mathrm{kk}} = g + a^2 \tilde{\omega}^2 \tag{2}$$

or equivalently, using the notation z for the points in P and the identification $z = (x, y) \in \Lambda \times \mathbb{R}$,

$$g^{\mathrm{kk}}(z)[w, w] = g^{\mathrm{kk}}(x, y)[(v, u), (v, u)] = g(x)[v, v] - a^2(u + \bar{\omega}(x)[v])^2,$$

for every $w = (v, u) \in T_x\Lambda \times \mathbb{R}$. The positive constant a has the dimension of a length and has been introduced for dimensional consistency of definition (2).

Let x_1 be an event in the chronological future of x_0. The set $\mathcal{N}_{x_0,x_1}$, includes the C^1 future-pointing non-spacelike connecting curves. With *connecting curve* we mean a map x from an interval $[a, b] \subset \mathbb{R}$ to Λ such that $x(a) = x_0$ and $x(b) = x_1$ and any other map w such that $w = x \circ \lambda$ with λ a C^1 function from an interval $[c, d]$ to the interval $[a, b]$, having positive derivative.

The functional $I[\gamma]$ defined on the space $\mathcal{N}_{x_0,x_1}$ is

$$I[\gamma](x_0, x_1) = \int_\gamma (\mathrm{d}s + \frac{q}{mc^2}\omega).$$

The timelike solutions of the Lorentz force equation (1), if they exists, are critical points of this functional as it follows from a computation of the Euler-Lagrange equation.

Let us now consider the geodesics over P. They are C^1 curves $z(\lambda) = (x(\lambda), y(\lambda))$ that are critical points of the functional

$$S = S(z) = \int_0^1 \frac{1}{2} g^{\mathrm{kk}}(z(\lambda))[\dot{z}(\lambda), \dot{z}(\lambda)]\,\mathrm{d}\,\lambda.$$

Taking into account that g^{kk} is independent of y we find that the following quantity is conserved

$$p_z = -a^2(\dot{y} + \bar{\omega}(x)[\dot{x}]).$$

Moreover taking variations with respect to the variable x we obtain

$$D_\lambda \dot{x} = p_z \hat{\bar{F}}(x)[\dot{x}]. \tag{3}$$

If x is non-spacelike we define

$$g(x)[\dot{x}, \dot{x}] = C^2.$$

Moreover, since z is a geodesic, $g^{\mathrm{kk}}(z)[\dot{z},\dot{z}]$ is conserved too and

$$g^{\mathrm{kk}}(z)[\dot{z},\dot{z}] = C^2 - \frac{p_z^2}{a^2}. \tag{4}$$

From this formula it follows that if z is timelike (non-spacelike) then x is timelike (non-spacelike). If z is a null geodesic then $C^2 = p_z^2/a^2$ and x is timelike iff $p_z \neq 0$.

In case x is timelike its proper time is given by

$$\mathrm{d}s = C\mathrm{d}\lambda,$$

and parameterizing with respect to proper time Eq. (3) becomes

$$D_s\left(\frac{\mathrm{d}\,x}{\mathrm{d}\,s}\right) = \frac{p_z}{C}\hat{\hat{F}}(x)\left[\frac{\mathrm{d}\,x}{\mathrm{d}\,s}\right] = \frac{p_z}{C}\frac{e}{\hbar c}\hat{F}(x)\left[\frac{\mathrm{d}\,x}{\mathrm{d}\,s}\right].$$

This is exactly the Lorentz force equation for a charge-to-mass ratio

$$\frac{q}{m} = \frac{p_z}{C}\frac{ec}{\hbar}.$$

Notice that, a solution of Eq. (3) must be timelike ($p_z \neq 0$) in order to represent a charged particle. Only in this case it can be parameterized with respect to proper time.

Our strategy is to search a future-directed null geodesic in P that projects on a connecting timelike curve on Λ. To this end we have to choose the following value for a

$$a = |\frac{p_z}{C}| = \frac{\hbar}{ec}|\frac{q}{m}|. \tag{5}$$

2. The theorem

We state the theorem.

Theorem 2.1. *Let (Λ, g) be a time-oriented Lorentzian manifold. Let ω be a one-form (C^2) on Λ and $F = \mathrm{d}\,\omega$. Assume that (Λ, g) is a globally hyperbolic manifold. Let x_1 be an event in the chronological future of x_0 and q/m any charge-to-mass ratio. There exists at least one future-directed non-spacelike C^1 curve $x(\lambda)$ connecting x_0 and x_1 that maximizes the functional $I[\gamma](x_0, x_1)$ on the space $\mathcal{N}_{x_0,x_1}$. If x is timelike, once parameterized with respect to proper time, it becomes a solution of the Lorentz force equation (1); if it is null, it is a null geodesic.*

We need some lemmas.

Lemma 2.2. *The manifold $P = \Lambda \times \mathbb{R}$ endowed with the metric* (2) *is a time-oriented globally hyperbolic Lorentzian manifold.*

Proof. See [5] or [4]. □

Remark 2.3. Let $E^+(p_0) = J^+(p_0) - I^+(p_0)$, $p_0 \in P$. It is well known (see [6, p. 112,184]) that if $q \in E^+(p_0)$ there exists a null geodesic connecting p_0 and q.

Lemma 2.4. *Any globally hyperbolic Lorentzian manifold Λ is causally simple, i.e. for every compact subset K of Λ, $\dot{J}^+(K) = E^+(K)$, where $\dot{J}^+(K)$ denotes the boundary of $J^+(K)$.*

Proof. See [6, p. 188, 207]. □

Proof of Theorem 2.1. Let P be the Kaluza-Klein principal bundle constructed in the introduction having a given by Eq. (5). Given a parameterized curve $\sigma(\lambda)$: $[0,1] \to \Lambda$ belonging to $\mathcal{N}_{x_0,x_1}$ define its lifts $\tilde{\sigma}^+(\lambda)$ and $\tilde{\sigma}^-(\lambda)$ of starting point $p_0 = (x_0, y_0)$ by requiring $p_{\tilde{\sigma}^\pm} = \pm a \int_\sigma \mathrm{d}s$ and $\tilde{\sigma}^\pm(0) = p_0$. In other words $\tilde{\sigma}^\pm(\lambda) = (\sigma(\lambda), y^\pm(\lambda))$ satisfies the condition

$$\dot{y}^\pm + \bar{\omega}[\dot{\sigma}] = -\frac{p_{\tilde{\sigma}^\pm}}{a^2}. \tag{6}$$

$\tilde{\sigma}^\pm(\lambda)$ is a null curve that depends on both σ and its parameterization. Let $y_1^\pm(\sigma) = \tilde{\sigma}^\pm(1)$ and $\Delta y^\pm(\sigma) = y_1^\pm(\sigma) - y_0$. Integrating Eq. (6) over σ

$$p_{\tilde{\sigma}^\pm} = -a^2(\Delta y^\pm(\sigma) + \int_\sigma \bar{\omega}),$$

or

$$\Delta y^\pm(\sigma) = y_1^\pm(\sigma) - y_0 = \mp\frac{1}{a}(\int_\sigma \mathrm{d}s + \frac{(\pm|q/m|)}{c^2}\int_\sigma \omega). \tag{7}$$

Notice that the final point $p_1 = (x_1, y_1^\pm)$ does not depend on the specific parameterization of σ. A maximization on $\mathcal{N}_{x_0,x_1}$ of the functional I relative to the ratio $+|q/m|$, corresponds to a minimization of $y_1^+(\sigma)$. Analogously, a maximization on $\mathcal{N}_{x_0,x_1}$ of the functional I relative to the ratio $-|q/m|$ corresponds to a maximization of y_1^-. Let

$$\begin{aligned} \hat{s} &= \sup_{\sigma\in\mathcal{N}_{x_0,x_1}} y_1^-(\sigma), \\ \bar{s} &= \inf_{\sigma\in\mathcal{N}_{x_0,x_1}} y_1^+(\sigma), \end{aligned}$$

we show that $\hat{s} > \bar{s}$. Indeed, for a given σ we have

$$y_1^-(\sigma) - y_1^+(\sigma) = \frac{2}{a}\int_\sigma \mathrm{d}s \geq 0 \tag{8}$$

thus

$$\hat{s} - \bar{s} \geq \frac{2}{a}\sup_{\sigma\in\ \mathcal{N}_{x_0,x_1}}\int_\sigma \mathrm{d}s = \frac{2l(x_0,x_1)}{a} \geq 0,$$

with $l(x_0, x_1)$ the Lorentzian distance function. Moreover, both $\int_\sigma \mathrm{d}s$ and $\int_\sigma |\bar{\omega}|$ are bounded on $\mathcal{N}_{x_0,x_1}$ [5], therefore $\hat{s}$ and $\bar{s}$ are finite.

Let $\eta : [0,1] \to P$ be a non-spacelike future-directed C^1 curve that starts in p_0 and ends in $p_1 : \pi(p_1) = x_1$. Let $p_1 = (x_1, y_1)$, and consider the projection $x(\lambda)$ of $\eta(\lambda)$. Since η in a non-spacelike curve

$$g(\dot{x},\dot{x}) - a^2(\dot{y} + \bar{\omega}(\dot{x}))^2 \geq 0.$$

Taking the square-root and integrating over $x(\lambda)$

$$|y_1 - y_0| \leq \frac{1}{a}\int_x \mathrm{d}s + \int_x |\bar{\omega}| < M < +\infty,$$

where M is a suitable positive constant. Hence y_1 is finite. Now we consider the set $W = J^+(p_0) \cap \pi^{-1}(x_0)$ and define

$$\begin{aligned} \hat{s}' &= \sup_{p\in W} y_1(p), \\ \bar{s}' &= \inf_{p\in W} y_1(p). \end{aligned}$$

where $y_1(p)$ is defined through $p=(x_1,y_1)$. Since for any non-spacelike curve y_1 is bounded, $\hat{s}'$ and $\bar{s}'$ are bounded too. Since P is globally hyperbolic $J^+(p_0)$ is closed and the set W, being limited and closed, is compact. The points $\hat{p}_1=(x_1,\hat{s}')$ and $\bar{p}_1=(x_1,\bar{s}')$, being accumulation points, belong to W. Moreover they can't be points of the open set $I^+(p_0)$. Thus, they belong to $E^+(p_0)$ and therefore (remark 2.3) there are two null geodesics $\hat{\eta}(\lambda)=(\hat{x}(\lambda),\hat{y}(\lambda))$, $\bar{\eta}(\lambda)=(\bar{x}(\lambda),\bar{y}(\lambda))$, that join p_0 with $\hat{p}_1$ and $\bar{p}_1$ respectively. Let λ be that affine parameter that has values 0 and 1 at the endpoints. For a null geodesic $\eta=(x,y)$ as those under consideration, p_η is conserved, hence for a suitable choice of sign $\eta=\tilde{x}^{\pm}$. Let $p_1=(x_1,y_1)$ be its final point. For the definition of $\hat{s}$ and $\bar{s}$

$$\bar{s}\le y_1\le \hat{s}.$$

But in the case $\eta=\hat{\eta}$ it is $\hat{s}'\ge\hat{s}$, otherwise there would be a null curve β having final point $\beta(1)$ strictly above $\hat{p}_1$ on x_1's fiber. This would be a contradiction since $\beta(1)\in W$ as β is a null curve. With an analogous reasoning for $\bar{\eta}$ we conclude that

$$\begin{aligned}\hat{s}' &= \hat{s},\\ \bar{s}' &= \bar{s}.\end{aligned}$$

We are going to show that the right choice of sign for $\hat{\eta}$ is $-$, that is

$$p_{\hat{\eta}}=-a\int_{\hat{x}}\mathrm{d}s\le 0,$$

and $\hat{\eta}=\tilde{\hat{x}}^-$.

Assume that $\hat{\eta}=\tilde{\hat{x}}^+$ then, from the definition of $\hat{s}'$ $(=\hat{s})$

$$\hat{s}'=\tilde{\hat{x}}^+(1)\ge\tilde{\hat{x}}^-(1).$$

Equation (8) implies that $\tilde{\hat{x}}^-(1)\ge\tilde{\hat{x}}^+(1)$ where the equality holds if and only if $\hat{x}$ is a null curve. From the hypothesis we find that $\hat{x}$ is a null curve and, since in this case both lifts coincide with the horizontal lift, $\tilde{\hat{x}}^-(\lambda)=\tilde{\bar{x}}^+(\lambda)$. Thus $-$ is always the right sign whereas $+$ is right if and only if $\hat{x}$ is a null curve, in which case $\tilde{\hat{x}}^-=\tilde{\hat{x}}^+$.

With an analogous reasoning for $\bar{\eta}$ we find

$$\begin{aligned}\hat{\eta}(\lambda) &= \tilde{\hat{x}}^-(\lambda),\\ \bar{\eta}(\lambda) &= \tilde{\bar{x}}^+(\lambda).\end{aligned}$$

We conclude that the functional $I[\gamma](x_0,x_1)$ is maximized in $\mathcal{N}_{x_0,x_1}$ by $\hat{x}$ if $q/m<0$ or by $\bar{x}$ if $q/m>0$. The curve $\hat{x}$, being the projection of a null geodesic, is a connecting solution of Eq. (3), moreover if timelike it is a connecting solution of the Lorentz force equation with charge-to-mass ratio $-|q/m|$. If it is null, from $|p_{\hat{\eta}}|=a\int_{\hat{x}}\mathrm{d}s=0$ and Eq. (3) we conclude that it is a null geodesic. An analogous conclusion holds for $\bar{x}$. □

In many cases the spacetime Λ has the property that no two chronologically related events are joined by a null geodesic. Minkowski spacetime is the most important example.

Corollary 2.5. *Let (M,η) be the Minkowski spacetime. Let F be an electromagnetic tensor field (closed two-form). Let x_1 be an event in the chronological future of x_0 and q/m a charge-to-mass ratio, then there exist at least one future-directed timelike solution to* (1) *connecting x_0 and x_1.*

Proof. Since M is contractible F is exact. Moreover, in Minkowski spacetime, if $x_1 \in I^+(x_0)$ there is no null geodesic connecting x_0 with x_1. $\square$

3. Conclusions

We have shown that in Minkowski spacetime the existence of at least a timelike connecting solution to the Lorentz force equation is assured by corollary 2.5. Notice that theorem 2.1 holds more generally for any chronologically related pair x_0, x_1, belonging to a globally hyperbolic set $N \subset M$. Thus in a generic spacetime the existence of a timelike connecting solution to the Lorentz force equation is assured whenever x_0 and x_1 belong to a globally hyperbolic set and there is no connecting null geodesic. Finally, we have proved the existence of at least one C^1 connecting curve that maximizes the functional $I[\gamma] = \int_\gamma \mathrm{d}s + q/(mc^2)\omega$ over the set of C^1 future-directed non-spacelike connecting curves.

References

[1] J. K. Beem, P. E. Ehrlich, K. L. Easley, *Global Lorentzian Geometry*, second edition. Marcel Dekker, Inc., New York and Basel, 1996.

[2] A. Avez, Essais edc géométrie Riemannienne hyperbolique globale. Application à la Relativité Générale, Ann. Inst. Fourier (Grenoble), **132** (1963) 105

[3] H.J. Seifert, Global connectivity by timelike geodesics, Zs. f. Naturfor. **22**a (1967) 1356

[4] E. Caponio, A. Masiello, The Avez-Seifert theorem for the relativistic Lorentz equation, preprint Rapporti DIM Università-Politecnico di Bari, 39/2001.

[5] E. Caponio, E. Minguzzi, Solutions to the Lorentz force equation with fixed charge-to-mass ratio in globally hyperbolic spacetimes, preprint gr-qc/0211100.

[6] S. W. Hawking, G. F. R. Ellis, *The Large Scale Structure of Space-Time*. Cambridge University Press, Cambridge, 1973.

[7] R. Kerner, J. Martin, S. Mignemi, J.-W. van Holten, Geodesic Deviation in Kaluza-Klein theories, Phys. Rev. D, **63** (2001) 027502.

[8] C. Misner, K. Thorne, J. Wheeler, *Gravitation*. W. H. Freeman, San Francisco, 1973.

[9] J. M. Overduin, P. S. Wesson, Kaluza-Klein Gravity, Phys.Rept. **283** (1997) 303

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